Semiclassical coherent state propagator for systems with spin

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Abstract

We derive the semiclassical limit of the coherent state propagator for systems with two degrees of freedom of which one degree of freedom is canonical and the other a spin. Systems in this category include those involving spin-orbit interactions and the Jaynes-Cummings model in which a single electromagnetic mode interacts with many independent two-level atoms. We construct a path integral representation for the propagator of such systems and derive its semiclassical limit. As special cases we consider separable systems, the limit of very large spins and the case of spin 1/2.

PACS numbers:

I. INTRODUCTION

The spin-orbit interaction plays important roles in many areas of physics, from atomic physics to condensed matter. The quantum description of systems with such interactions requires the use of Hilbert spaces which are the direct product of the orbital space (for which the coordinate or the momentum eigenstates form a basis) times the intrinsic space of the spin.

In quantum optics a similar situation arises in the study of the interaction between atoms and electromagnetic modes in a cavity. When only two states of the atoms are relevant, as for example in the ammonia maser, these two states can be formally represented by a spin 1/2. The state of a set of N such atoms can be likewise represented by the states of an angular momentum of larger magnitude. The Hamiltonian describing their interaction with a single electromagnetic mode of a cavity can therefore be written in terms of the operators $\hat{a}$ and $\hat{a}^\dagger$, which annihilate and create excitations of the quantized electromagnetic mode, and $\hat{J}_z$, $\hat{J}_+$ and $\hat{J}_-$, of the angular momentum.

The semiclassical behavior of such systems has drawn attention for quite a long time. One natural representation for the study of this limit is that of coherent states. The semiclassical limit of the coherent state propagator for both the Weyl and the SU(2) group has already been studied in detail. The purpose of this paper is to derive the semiclassical limit of the coherent state propagator for general systems with two degrees of freedom in which one degree is canonical and the other a spin.

The quantum propagator $K(b''^*, b', T) \equiv \langle b''|e^{-i\hat{H}T/\hbar}|b'\rangle$ represents the probability amplitude that the initial state $|b'\rangle$ be measured as $|b''\rangle$, after a time T, when evolved by the Hamiltonian $\hat{H}$. The propagator is the essential ingredient for quantum dynamical calculations and it is also fundamental in the study of the quantum-classical connection. Semiclassical approximations of the propagator in the coordinates and momentum representations were studied initially by Feynman [1] and later by many others [2]. Semiclassical formulae for the propagator in the basis of coherent states appeared for the first time in the works of Klauder [3, 4]. Although these papers have treated the propagator for both canonical coherent states $|z\rangle$ and spin coherent states $|s\rangle$, not much work has been done on these two basis simultaneously. The development of the semiclassical theory occurred independently on each representation.

For canonical coherent states, Weissman [5, 6] re-derived the results of Klauder using the semiclassical theory of Miller [7]. In Weissman's work, and also in the original Klauder's papers, the fluctuations around the critical trajectory have not been accurately performed, and a 'phase' factor was missed. In spite of this, a first numerical application of the semiclassical formula was performed by Adachi [8] for an two-dimensional chaotic map, obtaining good agreement with exact quantum results. The correct evaluation of the second order fluctuations appeared in the works of Baranger and Aguiar [9], Xavier and Aguiar [10, 11, 12] and, independently, by Kochetov [13]. However, a detailed derivation of the semiclassical coherent state propagator for one-dimensional systems has appeared only later in [14]. Extensions of the formula for two dimensional degrees of freedom and applications to chaotic systems have been performed in Ref. [15].

There are two differences between the semiclassical formula of Baranger *et al* and that of Klauder and Weissman. First, there is the extra 'phase' in the new formula, which is usually complex and, in fact, is a signature of the basis of coherent states. It is related to the overcompletness of the basis, since changing the resolution of the unity also changes this term [14, 16]. A second difference consists in replacing the Weyl symbol H of the Hamiltonian operator $\hat{H}$ by the average $\tilde{H}_z \equiv \langle z|\hat{H}|z\rangle$. This implies that the classical trajectories entering in the formula are subjected to $\tilde{H}_z$ instead of H. The dynamics with $\tilde{H}$ actually appears naturally in the work by Klauder, but it was changed back to H to be consistent with the lack of the extra exponential factor of the formula. As discussed in Ref. [14], both these changes are essential to get good agreement between quantum and semiclassical results. We note that the semiclassical coherent state propagator also presents singularities and discontinuities due to phase space caustics and the Stokes phenomenon [8, 15, 17, 18, 19].

Approximations for spin coherent states have also been studied by Kuratsuji and Suzuki [20]. The basic difference between their approach and Klauder's is that the later represents the classical trajectories in a Bloch sphere by means of angle variables while Kuratsuji and Suzuki represents the same dynamics in terms of an stereographic projection of the Bloch sphere on a complex plane. As is the case of canonical coherent states, a detailed derivation in the spin coherent state propagator only appeared later with Solari [21], where features similar to those appearing in the canonical case were found: an extra exponential term, and the underlying classical dynamics governed by the average Hamiltonian $\tilde{H}_s \equiv \langle s|\hat{H}|s\rangle$. Vieira

and Sacramento [22] and Kochetov [23] have also derived the same formula independently. Yet another detailed derivation has also been presented by Stone *et al* [24], focusing on the importance of the extra term (see also Ref. [25]), which has received the name of *Solari-Kochetov phase*. The singularities and discontinuities of the semiclassical spin propagator are discussed in Refs. [26, 27].

In spite of the extensive work on the semiclassical theory for the canonical and spin representations, there are very few results considering the two bases simultaneously. Alscher and Grabert [28] have calculated the semiclassical coherent state propagator for the spin 1/2 Jaynes-Cummings model showing that it corresponds to the exact quantum result. Pletyukhov et al [29, 30] derived a semiclassical trace formula for systems with spin, but did not consider the semiclassical propagator itself in detail. An overview of the semiclassical approaches for spin-orbit interactions and associated trace formulae was recently published by Amann and Brack [31], and a general semiclassical theory for Hamiltonians which are linear in spin operators has also been formulated [29, 30, 32, 33].

In this paper we derive a semiclassical formula for general canonical-spin systems using path integrals. We show that our formula agrees with the previous results [14, 21, 22, 23] in the separable case (no spin-orbit interaction) and that it reduces to the 2-D canonical coherent state propagator in the limit of very large spins. Finally we discuss the limit of validity of the approximation for spin-$\frac{1}{2}$ systems.

II. BASIC DEFINITIONS

The propagator in the canonical-spin representation is given by

$$K\left(z''^{*}, s''^{*}, z', s', T\right) = \langle z'', s''|e^{-i\hat{H}T/\hbar}|z', s'\rangle, \tag{1}$$

where $|z, s\rangle \equiv |z\rangle \otimes |s\rangle$ is the product of a canonical coherent state $|z\rangle$ and a spin coherent state $|s\rangle$. While $|z\rangle$ is defined in the 'particle' Hilbert space, $|s\rangle$ is defined in the $(2j+1)$-dimensional space of an angular momentum j. These states can be written as

$$|z\rangle = e^{z\hat{a}^{\dagger} - \frac{1}{2}|z|^2}|0\rangle \qquad \text{and} \qquad |s\rangle = \frac{e^{s\hat{J}_+}}{(1+|s|^2)^j}|-j\rangle, \tag{2}$$

where z and s are complex numbers, $\hat{a}^{\dagger}$ is the canonical creator operator, $\hat{J}_+$ is the raising spin operator, $|0\rangle$ is the harmonic oscillator vacuum state and $|-j\rangle$ is the extremal eigenstate

of $\hat{J}_3$ with eingenvalue $-j$. The coherent states are non-orthogonal with

$$\langle z_1|z_2\rangle = \exp\left\{-\frac{1}{2}|z_1|^2 + z_1^* z_2 - \frac{1}{2}|z_2|^2\right\} \qquad \text{and} \qquad \langle s_1|s_2\rangle = \frac{(1+s_1^* s_2)^{2j}}{(1+|s_1|^2)^j\,(1+|s_2|^2)^j}, \quad (3)$$

and satisfy

$$\int \frac{\mathrm{d}x\,\mathrm{d}y}{\pi}|z\rangle\langle z| \equiv 1_{(z)} \qquad \text{and} \qquad \frac{2j+1}{\pi}\int \frac{\mathrm{d}X\,\mathrm{d}Y}{(1+|s|^2)^2}|s\rangle\langle s| \equiv 1_{(s)}, \tag{4}$$

where x and y are the real and imaginary parts of z and X and Y the real and imaginary parts of s. Throughout this paper we shall use lower case letters to refer to the canonical variables and corresponding upper case letters to refer to the spin variables.

Finally, the complex number z labelling the canonical coherent state can be written explicitly in terms of position and momentum variables as

$$z = \frac{1}{\sqrt{2}}\left(\frac{q}{b} + i\frac{p}{c}\right) \tag{5}$$

where q and p are the average values of the position and momentum operators, respectively, and the variances b and c satisfy $bc = \hbar$.

III. PATH INTEGRAL FORMULATION

In this section we construct a path integral representation for the quantum propagator (1). As usual we divide the time T into N small intervals of size $\epsilon = T/N$ and insert resolutions of unity between each propagation step. We obtain

$$\begin{aligned} K\left(z''^*, s''^*, z', s', T\right) &= \lim_{\epsilon\to 0}\int \prod_{k=1}^{N-1}\left\{\left(\frac{2j+1}{\pi^2}\right)\frac{\mathrm{d}x_k \mathrm{d}y_k \mathrm{d}X_k \mathrm{d}Y_k}{(1+|s_k|^2)^2}\right\} \\ &\times \prod_{k=0}^{N-1}\langle z_{k+1}, s_{k+1}|e^{-i\hat{H}\epsilon/\hbar}|z_k, s_k\rangle, \end{aligned} \tag{6}$$

where we define $|z_0, s_0\rangle \equiv |z', s'\rangle$ and $\langle z_N, s_N| \equiv \langle z'', s''|$. In the limit $\epsilon \to 0$ the infinitesimal propagators can be written as

$$\langle z_{k+1}, s_{k+1}|e^{-i\hat{H}\epsilon/\hbar}|z_k, s_k\rangle = \langle z_{k+1}, s_{k+1}|z_k, s_k\rangle \exp\left\{-\frac{i}{\hbar}\epsilon\tilde{H}_{k+1/2}\right\}, \tag{7}$$

where

$$\tilde{H}_{k+1/2} \equiv \frac{\langle z_{k+1}, s_{k+1}|\hat{H}|z_k, s_k\rangle}{\langle z_{k+1}, s_{k+1}|z_k, s_k\rangle} \tag{8}$$

and

$$\langle z_{k+1}, s_{k+1}|z_k, s_k\rangle = \frac{\left(1+s_{k+1}^* s_k\right)^{2j}}{\left(1+|s_{k+1}|^2\right)^j\left(1+|s_k|^2\right)^j} \exp\left\{-\frac{1}{2}|z_{k+1}|^2 + z_{k+1}^* z_k - \frac{1}{2}|z_k|^2\right\}. \quad (9)$$

With these considerations the propagator can be written as

$$K\left(z''^*, s''^*, z', s', T\right) = \lim_{\epsilon\to 0}\left(\frac{2j+1}{\pi^2}\right)^{N-1} \int \prod_{k=1}^{N-1}\left\{\mathrm{d}x_k \mathrm{d}y_k \mathrm{d}X_k \mathrm{d}Y_k\right\} e^F, \quad (10)$$

where

$$F = F_z + F_s - \frac{i}{\hbar}\sum_{k=0}^{N-1} \epsilon \tilde{H}_{k+1/2}, \quad (11)$$

with

$$F_z = \frac{i}{\hbar}\sum_{k=0}^{N-1} \frac{i\hbar}{2}\left[z_k z_k^* - 2 z_{k+1}^* z_k + z_{k+1}^* z_{k+1}\right] \quad (12)$$

and

$$F_s = \frac{i}{\hbar}\left\{-i\hbar j \sum_{k=0}^{N-1} \ln\left[\frac{\left(1+s_{k+1}^* s_k\right)^2}{\left(1+s_k^* s_k\right)\left(1+s_{k+1}^* s_{k+1}\right)}\right] - 2i\hbar \sum_{k=1}^{N-1} \ln\left[\frac{1}{\left(1+s_k^* s_k\right)}\right]\right\}. \quad (13)$$

Eq. (10) is a discretized path integral representation of the propagator (1). In the following sections we shall consider the formal semiclassical limit $\hbar \to 0$ and $j \to \infty$, keeping the product $j\hbar \equiv J$ constant.

IV. THE SEMICLASSICAL LIMIT

In the semiclassical limit, the integrals in Eq. (10) can be performed by the saddle point method. The method consists basically in approximating the exponent F by a quadratic form and performing the resulting Gaussian integrals. The quadratic form is obtained by expanding F around its critical points. In the next subsections we shall: (a) find the critical points of F, and therefore the critical path; (b) compute F at the critical path; (c) expand F to second order around the critical path and compute the Gaussian integrals.

A. The Critical Path

We begin by looking for critical points of F. They satisfy the equations

$$\frac{\partial F}{\partial z_m} = \frac{\partial F}{\partial z_m^*} = \frac{\partial F}{\partial s_m} = \frac{\partial F}{\partial s_m^*} = 0, \qquad m = 1, \ldots, N-1. \quad (14)$$

As $\tilde{H}_{k+1/2} = \tilde{H}_{k+1/2}(z^*_{k+1}, s^*_{k+1}, z_k, s_k)$, these equations can be explicitly written as

$$\begin{aligned}\frac{i\epsilon}{\hbar}\frac{\partial \tilde{H}_{m+1/2}}{\partial z_m} &= z^*_{m+1} - z^*_m,\\ \frac{i\epsilon}{\hbar}\frac{\partial \tilde{H}_{m-1/2}}{\partial z^*_m} &= z_{m-1} - z_m,\end{aligned} \tag{15}$$

and

$$\begin{aligned}\frac{i\epsilon}{2\hbar}\frac{\partial \tilde{H}_{m+1/2}}{\partial s_m} &= j\left\{\frac{s^*_{m+1}}{1+s^*_{m+1}s_m} - \frac{s^*_m}{1+s^*_m s_m}\right\} - \frac{s^*_m}{1+s^*_m s_m},\\ \frac{i\epsilon}{2\hbar}\frac{\partial \tilde{H}_{m-1/2}}{\partial s^*_m} &= -j\left\{\frac{s_m}{1+s^*_m s_m} - \frac{s_{m-1}}{1+s^*_m s_{m-1}}\right\} - \frac{s_m}{1+s^*_m s_m}.\end{aligned} \tag{16}$$

It is important to emphasize that, because $m = 1, \ldots, N-1$, the variables z^*_0, s^*_0, z_N, s_N do not enter in Eqs. (15) and (16): the critical path, defined by the set of critical points, depends only on z_0, s_0, z^*_N and s^*_N, and not on z^*_0, s^*_0, z_N, s_N.

In the limit $\epsilon \to 0$ Eqs. (15) become

$$\frac{i}{\hbar}\frac{\partial \tilde{H}}{\partial z} = \dot{z}^* \qquad \text{and} \qquad \frac{i}{\hbar}\frac{\partial \tilde{H}}{\partial z^*} = -\dot{z} \tag{17}$$

where $\tilde{H} \equiv \langle z, s|\hat{H}|z, s\rangle$. In terms of q and p (see Eq.(5)) this corresponds to the usual Hamilton's equations

$$\frac{\partial \tilde{H}}{\partial p} = \dot{q} \qquad \text{and} \qquad \frac{\partial \tilde{H}}{\partial q} = -\dot{p} \tag{18}$$

For Eqs. (16) the calculation is slightly more involved but the result is also very simple. We obtain, in the limit $\epsilon \to 0$ and $\hbar \to 0$ with $j = J/\hbar$,

$$\frac{\partial \tilde{H}}{\partial s} = -\frac{2iJ\dot{s}^*}{(1+s^*s)^2} \qquad \text{and} \qquad \frac{\partial \tilde{H}}{\partial s^*} = \frac{2iJ\dot{s}}{(1+s^*s)^2}. \tag{19}$$

The trajectories described by Eqs. (17) and (19) define the critical path of the Feynman integral (10). Nevertheless, these trajectories must satisfy the boundary conditions $z(0) = z'$, $s(0) = s'$, $z^*(T) = z''^*$ and $s^*(T) = s''^*$, as can be seen from Eqs. (15) and (16). As discussed in detail in [14], these trajectories are usually complex and the variables z and z^* are not generally the complex conjugate of each other, the same happening between s and s^*. Therefore, it is convenient to set a new notation

$$z = u, \qquad s = U, \qquad z^* = v \qquad \text{and} \qquad s^* = V. \tag{20}$$

In terms of these new variables, the equations of motion (17) and (19) become

$$\frac{i}{\hbar}\frac{\partial \tilde{H}}{\partial u} = \dot{v}, \qquad \frac{i}{\hbar}\frac{\partial \tilde{H}}{\partial v} = -\dot{u}, \qquad \frac{i}{\hbar}\frac{\partial \tilde{H}}{\partial U} = \frac{2j\dot{V}}{(1+UV)^2}, \qquad \frac{i}{\hbar}\frac{\partial \tilde{H}}{\partial V} = \frac{-2j\dot{U}}{(1+UV)^2}, \tag{21}$$

with boundary conditions

$$u(0) = z', \qquad U(0) = s', \qquad v(T) = z''^*, \qquad V(T) = s''^*. \tag{22}$$

Since z'^*, s'^*, z'' and s'' do not appear in the boundary conditions, the value of $u(T)$, $U(T)$, $v(0)$ and $V(0)$ are determined by the integration of Eqs. (21). From now on we shall use this new notation to refer to the complex classical trajectories. The discrete variables z_m, s_m, z_m^* and s_m^* shall also be replaced by u^m, U^m, v^m and V^m, respectively.

B. The Complex Action

The function F appearing in Eq. (11) can now be calculated at the classical trajectory. We use a bar over the variables to indicate that they are calculated at the critical path. We have

$$\begin{aligned} F &= \frac{i}{\hbar}\sum_{k=0}^{N-1}\left\{\frac{i\hbar}{2}\left[\bar{u}^k\bar{v}^k - 2\bar{v}^{k+1}\bar{u}^k + \bar{v}^{k+1}\bar{u}^{k+1}\right] - i\hbar j \ln\left[\frac{\left(1+\bar{V}^{k+1}\bar{U}^k\right)^2}{\left(1+\bar{V}^k\bar{U}^k\right)\left(1+\bar{V}^{k+1}\bar{U}^{k+1}\right)}\right]\right\} \\ &- \frac{i}{\hbar}\sum_{k=0}^{N-1}\epsilon\tilde{H}_{k+1/2}(\bar{v}^{k+1},\bar{V}^{k+1},\bar{u}^k,\bar{U}^k) - 2\sum_{k=1}^{N-1}\ln\left(1+\bar{V}^k\bar{U}^k\right) \\ &- \frac{1}{2}\left(z'z'^* + z''z''^* - \bar{u}^0\bar{v}^0 - \bar{u}^N\bar{v}^N\right) - j\ln\left[\frac{(1+s's'^*)(1+s''s''^*)}{(1+\bar{U}^0\bar{V}^0)(1+\bar{U}^N\bar{V}^N)}\right]. \end{aligned}$$

As usual we have replaced z'^*, s'^*, z'', s'' by $\bar{v}^0, \bar{V}^0, \bar{u}^N, \bar{v}^N$ in the first line and corrected for this in the last line. Taking the limit $\epsilon \to 0$ we find, after some algebra,

$$F = \frac{i}{\hbar}\mathcal{S}(z''^*, s''^*, z', s', T) - \Lambda - 2\sum_{k=1}^{N-1}\ln\left(1+\bar{V}^k\bar{U}^k\right), \tag{23}$$

where $\mathcal{S}(z''^*, s''^*, z', s', T)$ is the complex action

$$\begin{aligned} \mathcal{S}(z''^*, s''^*, z', s', T) = &\int_0^T \left\{\frac{i\hbar}{2}\left(\dot{\bar{u}}\bar{v} - \dot{\bar{v}}\bar{u}\right) - i\hbar j\left(\frac{\bar{U}\dot{\bar{V}} - \bar{V}\dot{\bar{U}}}{1+\bar{U}\bar{V}}\right) - \tilde{H}\right\} dt \\ &- \frac{i\hbar}{2}\left(\bar{u}'\bar{v}' + \bar{u}''\bar{v}''\right) - i\hbar j \ln\left[(1+\bar{U}'\bar{V}')(1+\bar{U}''\bar{V}'')\right] \end{aligned} \tag{24}$$

and

$$\Lambda = \frac{1}{2}\left(|z'|^2 + |z''|^2\right) + j \ln\left[(1+|s'|^2)(1+|s''|^2)\right] \tag{25}$$

is a 'normalization term'. The limit $\epsilon \to 0$ has not been taken on last term of Eq.(23) because this term is going to get cancelled later on.

It can be checked that

$$\frac{\partial \mathcal{S}}{\partial \bar{u}'} = -i\hbar \bar{v}', \qquad \frac{\partial \mathcal{S}}{\partial \bar{v}''} = -i\hbar \bar{u}'', \tag{26}$$

$$\frac{\partial \mathcal{S}}{\partial \bar{U}'} = \frac{-2i\hbar j \bar{V}'}{1+\bar{U}'\bar{V}'}, \qquad \frac{\partial \mathcal{S}}{\partial \bar{V}''} = \frac{-2i\hbar j \bar{U}''}{1+\bar{V}''\bar{U}''}, \tag{27}$$

and

$$\frac{\partial \mathcal{S}}{\partial T} = -\tilde{H}, \tag{28}$$

where a single (double) prime to denotes initial (final) time.

C. Fluctuations around the Critical Path

In the semiclassical limit, the only relevant points in the path integral of Eq. (10) are the saddle points. In the present case, they define trajectories governed by Eqs. (21) [or by their discretized forms, Eqs. (15) and (16)]. The exponent F in Eq. (10) has to be integrated over the intermediate points $\mathbf{x} \equiv (u^1, U^1, v^1, V^1, \ldots, u^{N-1}, U^{N-1}, v^{N-1}, V^{N-1})^T$. Expanding F around the critical trajectory $\bar{\mathbf{x}}$ we obtain

$$F(\mathbf{x}) = F(\bar{\mathbf{x}}) - \frac{1}{2}\delta\mathbf{x}^T[-\delta^2 F(\bar{\mathbf{x}})]\delta\mathbf{x}, \tag{29}$$

where $\delta\mathbf{x} \equiv \mathbf{x} - \bar{\mathbf{x}}$. The matrix $[-\delta^2 F(\bar{\mathbf{x}})]$ contains the second derivatives of F calculated with the critical trajectory. Substituting this result in Eq. (10) and considering the jacobian

$$\{\mathrm{d}x_k \mathrm{d}y_k \mathrm{d}X_k \mathrm{d}Y_k\} \to -\frac{1}{4}\{\mathrm{d}z_k \mathrm{d}z_k^* \mathrm{d}s_k \mathrm{d}s_k^*\} \equiv -\frac{1}{4}\left\{\mathrm{d}u^k \mathrm{d}v^k \mathrm{d}U^k \mathrm{d}V^k\right\},$$

we find

$$K = e^{F(\bar{\mathbf{x}})} \lim_{\epsilon \to 0}\left\{\left(\frac{2j+1}{-4\pi^2}\right)^{N-1} \int \prod_{k=1}^{N-1}\left\{\mathrm{d}[\delta u^k]\mathrm{d}[\delta v^k]\mathrm{d}[\delta U^k]\mathrm{d}[\delta V^k]\right\} e^{-\frac{1}{2}\delta\mathbf{x}^T[-\delta^2 F(\bar{\mathbf{x}})]\delta\mathbf{x}}\right\}, \tag{30}$$

where $F(\bar{\mathbf{x}})$ is given by Eq. (23). The matrix $[-\delta^2 F(\bar{\mathbf{x}})]$ is written as

$$\left(\begin{array}{cccc|cccc|c}
\mathcal{H}_{11}^{N-1} & \mathcal{H}_{21}^{N-1} & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\
\mathcal{H}_{21}^{N-1} & \mathcal{H}_{22}^{N-1} & 0 & B^{N-2} & 0 & 0 & 0 & 0 & \dots \\
1 & 0 & \mathcal{H}_{33}^{N-2} & \mathcal{H}_{43}^{N-2} & \mathcal{H}_{13}^{N-2}-1 & \mathcal{H}_{23}^{N-2} & 0 & 0 & \dots \\
0 & B^{N-2} & \mathcal{H}_{43}^{N-2} & \mathcal{H}_{44}^{N-2} & \mathcal{H}_{41}^{N-2} & \mathcal{H}_{24}^{N-2}-\mathcal{B}^{N-2} & 0 & 0 & \dots \\
\hline
0 & 0 & \mathcal{H}_{31}^{N-2}-1 & \mathcal{H}_{41}^{N-2} & \mathcal{H}_{11}^{N-2} & \mathcal{H}_{21}^{N-2} & 1 & 0 & \dots \\
0 & 0 & \mathcal{H}_{23}^{N-2} & \mathcal{H}_{42}^{N-2}-\mathcal{B}^{N-2} & \mathcal{H}_{21}^{N-2} & \mathcal{H}_{22}^{N-2} & 0 & B^{N-3} & \dots \\
0 & 0 & 0 & 0 & 1 & 0 & \mathcal{H}_{33}^{N-3} & \mathcal{H}_{43}^{N-3} & \dots \\
0 & 0 & 0 & 0 & 0 & B^{N-3} & \mathcal{H}_{43}^{N-3} & \mathcal{H}_{44}^{N-3} & \dots \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right),$$

where

$$B^m = \frac{2(j+1)}{(1+\bar{V}^m\bar{U}^m)^2} \quad , \qquad \mathcal{B}^m = \frac{2j}{(1+\bar{V}^{m+1}\bar{U}^m)^2} \tag{31}$$

and $\mathcal{H}_{ij}^m \equiv \partial^2\tilde{\mathcal{H}}_{m+1/2}/\partial\chi_i^{m_i}\partial\chi_j^{m_j}$. In this last definition we are using the compact notation

$$\chi_1 \equiv u, \qquad \chi_2 \equiv U, \qquad \chi_3 \equiv v \qquad \text{and} \qquad \chi_4 \equiv V. \tag{32}$$

In addition, m_i (m_j) equals to $m+1$ when i (j) is equal to 3 or 4, and equals to m when i (j) is equal to 1 or 2. In order to cancel the last term in Eq.(23) and also the factor $(2j+1)^{N-1}$ in the pre-factor of Eq.(30) we change the variables associated with the spin by the transformation $\delta U^m = \delta\tilde{U}^m/\sqrt{B^m}$ and $\delta V^m = \delta\tilde{V}^m/\sqrt{B^m}$. Eq. (30) then becomes

$$K = e^{\frac{i}{\hbar}\mathcal{S}(\bar{\mathbf{x}})-\Lambda}\lim_{\epsilon\to 0}\left\{\left(\frac{-1}{4\pi^2}\right)^{N-1}\int\prod_{k=1}^{N-1}\left\{\mathrm{d}[\delta u^k]\mathrm{d}[\delta v^k]\mathrm{d}[\delta\tilde{U}^k]\mathrm{d}[\delta\tilde{V}^k]\right\}e^{-\frac{1}{2}\delta\tilde{\mathbf{x}}^T[-\delta^2\tilde{F}(\bar{\mathbf{x}})]\delta\tilde{\mathbf{x}}}\right\}, \tag{33}$$

where $\mathcal{S}(\bar{\mathbf{x}})$ and Λ are given by Eqs. (24) and (25) respectively and $\delta\tilde{\mathbf{x}} \equiv (\delta u^1, \delta\tilde{U}^1, \delta v^1, \delta\tilde{V}^1, \dots, \delta u^{N-1}, \delta\tilde{U}^{N-1}, \delta v^{N-1}, \delta\tilde{V}^{N-1})^T$ and $[-\delta^2\tilde{F}(\bar{\mathbf{x}})]$ is the matrix

$$-\delta^2\tilde{F}(\bar{\mathbf{x}}) = \begin{pmatrix}
\mathrm{W}^{N-1} & \mathrm{R}^{N-1} & 0 & \dots \\
\left(\mathrm{R}^{N-1}\right)^T & \mathrm{W}^{N-2} & \mathrm{R}^{N-2} & \dots \\
0 & \left(\mathrm{R}^{N-2}\right)^T & \mathrm{W}^{N-3} & \dots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}, \tag{34}$$

where

$$
\mathrm{W}^k = \begin{pmatrix} \mathcal{H}_{11}^k & b^k\mathcal{H}_{21}^k & 1 & 0 \\ b^k\mathcal{H}_{21}^k & (b^k)^2\mathcal{H}_{22}^k & 0 & 1 \\ 1 & 0 & \mathcal{H}_{33}^{k-1} & b^k\mathcal{H}_{43}^{k-1} \\ 0 & 1 & b^k\mathcal{H}_{43}^{k-1} & (b^k)^2\mathcal{H}_{44}^{k-1} \end{pmatrix}
$$

and

$$
\mathrm{R}^k = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathcal{H}_{13}^{k-1} - 1 & b^{k-1}\mathcal{H}_{23}^{k-1} & 0 & 0 \\ b^k\mathcal{H}_{41}^{k-1} & b^k b^{k-1}(\mathcal{H}_{24}^{k-1} - \mathcal{B}^{k-1}) & 0 & 0 \end{pmatrix}.
$$

The matrix $(\mathrm{R}^m)^T$ is the transpose of R^m and $b^m \equiv 1/\sqrt{B^m}$.

The Gaussian integral in Eq. (33) can be solved immediately and the result is

$$
\int \prod_{k=1}^{N-1} \left\{ \mathrm{d}[\delta u^k]\mathrm{d}[\delta v^k]\mathrm{d}[\delta \tilde{U}^k]\mathrm{d}[\delta \tilde{V}^k] \right\} \exp\left\{ -\frac{1}{2}\delta\tilde{\mathbf{x}}^T[-\delta^2\tilde{F}(\bar{\mathbf{x}})]\delta\tilde{\mathbf{x}} \right\} = \sqrt{\frac{(2\pi)^{4(N-1)}}{\det[-\delta^2\tilde{F}(\bar{\mathbf{x}})]}}. \quad (35)
$$

The evaluation of the fluctuation determinant is the most lengthy step of the whole calculation. Here we shall only briefly describe the main steps of the calculation.

We call Δ^N the determinant of $[-\delta^2\tilde{F}(\bar{\mathbf{x}})]$ and we solve it by the Laplace method of reducing it to smaller determinants. In this process we are lead to define 5 auxiliary matrices whose determinants, together with Δ^N, form a closed set of six discrete recurrence relations. The five determinants are called Δ_0^N and Δ_{ij}^N, for $i, j = 1$ or 2. The former is the determinant of $[-\delta^2\tilde{F}(\bar{\mathbf{x}})]$ without the first two lines and columns, while Δ_{ij}^N is the determinant of $[-\delta^2\tilde{F}(\bar{\mathbf{x}})]$ without the first, second and i-th lines and without the first, second and j-th columns. Taking the limit of $N \to \infty$ in the mentioned set of relations we obtain the following set of linear differential equations:

$$
\dot{\mathbf{D}} = \frac{i}{\hbar} \begin{pmatrix} 0 & -\mathbf{H}_{22} & -\mathbf{H}_{11} & -\mathbf{H}_{21} & -\mathbf{H}_{21} & 0 \\ \mathbf{H}_{44} & -2\mathbf{H}_{24} & 0 & -\mathbf{H}_{41} & -\mathbf{H}_{41} & -\mathbf{H}_{11} \\ \mathbf{H}_{33} & 0 & -2\mathbf{H}_{13} & -\mathbf{H}_{23} & -\mathbf{H}_{23} & -\mathbf{H}_{22} \\ \mathbf{H}_{43} & -\mathbf{H}_{23} & -\mathbf{H}_{41} & -\mathbf{H}_{+} & 0 & \mathbf{H}_{21} \\ \mathbf{H}_{43} & -\mathbf{H}_{23} & -\mathbf{H}_{41} & 0 & -\mathbf{H}_{+} & \mathbf{H}_{21} \\ 0 & \mathbf{H}_{33} & \mathbf{H}_{44} & -\mathbf{H}_{43} & -\mathbf{H}_{43} & -2\mathbf{H}_{+} \end{pmatrix} \mathbf{D} \quad (36)
$$

where $\mathbf{D}^T(t) = (\Delta(t), \Delta_{11}(t), \Delta_{22}(t), \Delta_{12}(t), \Delta_{21}(t), \Delta_0(t))$ is a vector containing the original determinant $\Delta(t)$ and the five auxiliary ones. $\mathbf{H}_{ij}$ are elements of the matrix

$$\mathbf{H} \equiv \begin{pmatrix} \frac{\partial^2 \tilde{H}}{\partial u \partial u} & d\frac{\partial^2 \tilde{H}}{\partial u \partial U} & \frac{\partial^2 \tilde{H}}{\partial u \partial v} & d\frac{\partial^2 \tilde{H}}{\partial u \partial V} \\ d\frac{\partial^2 \tilde{H}}{\partial U \partial u} & d^2\left[\frac{\partial^2 \tilde{H}}{\partial U \partial U} + \frac{2V}{1+UV}\frac{\partial \tilde{H}}{\partial U}\right] & d\frac{\partial^2 \tilde{H}}{\partial U \partial v} & d^2\left[\frac{\partial^2 \tilde{H}}{\partial U \partial V} + \frac{V\frac{\partial \tilde{H}}{\partial V} + U\frac{\partial \tilde{H}}{\partial U}}{1+UV}\right] \\ \frac{\partial^2 \tilde{H}}{\partial v \partial u} & d\frac{\partial^2 \tilde{H}}{\partial v \partial U} & \frac{\partial^2 \tilde{H}}{\partial v \partial v} & d\frac{\partial^2 \tilde{H}}{\partial v \partial V} \\ d\frac{\partial^2 \tilde{H}}{\partial V \partial u} & d^2\left[\frac{\partial^2 \tilde{H}}{\partial U \partial V} + \frac{V\frac{\partial \tilde{H}}{\partial V} + U\frac{\partial \tilde{H}}{\partial U}}{1+UV}\right] & d\frac{\partial^2 \tilde{H}}{\partial V \partial v} & d^2\left[\frac{\partial^2 \tilde{H}}{\partial V \partial V} + \frac{2U}{1+UV}\frac{\partial \tilde{H}}{\partial V}\right] \end{pmatrix},$$

calculated with the classical trajectory where $d = (1 + \bar{U}\bar{V})/\sqrt{2j}$ and $\mathbf{H}_\pm \equiv \mathbf{H}_{13} \pm \mathbf{H}_{24}$.

Setting $\mathbf{D'} = \mathbf{D}e^{-\frac{i}{\hbar}\int \mathbf{H}_+ dt}$ we obtain the more symmetric form

$$\dot{\mathbf{D}}' = \frac{i}{\hbar}\mathcal{A}\,\mathbf{D}' \tag{37}$$

with

$$\mathcal{A} = \begin{pmatrix} \mathbf{H}_+ & -\mathbf{H}_{22} & -\mathbf{H}_{11} & -\mathbf{H}_{21} & -\mathbf{H}_{21} & 0 \\ \mathbf{H}_{44} & \mathbf{H}_- & 0 & -\mathbf{H}_{41} & -\mathbf{H}_{41} & -\mathbf{H}_{11} \\ \mathbf{H}_{33} & 0 & -\mathbf{H}_- & -\mathbf{H}_{23} & -\mathbf{H}_{23} & -\mathbf{H}_{22} \\ \mathbf{H}_{43} & -\mathbf{H}_{23} & -\mathbf{H}_{41} & 0 & 0 & \mathbf{H}_{21} \\ \mathbf{H}_{43} & -\mathbf{H}_{23} & -\mathbf{H}_{41} & 0 & 0 & \mathbf{H}_{21} \\ 0 & \mathbf{H}_{33} & \mathbf{H}_{44} & -\mathbf{H}_{43} & -\mathbf{H}_{43} & -\mathbf{H}_+ \end{pmatrix}. \tag{38}$$

The equations for $\mathbf{D'}$ are intimately related to the linearized equations of motion around the critical trajectory. To see this we go back to our notation as is Eqs. (32). Setting the small displacements $\delta\chi_i(t) \equiv \chi_i(t) - \bar{\chi}_i(t)$ around the critical trajectory and defining

$$\xi_1(t) = \delta u(t), \qquad \xi_2(t) = \frac{\sqrt{2j}}{(1 + U(t)V(t))}\delta U(t), \tag{39}$$

$$\xi_3(t) = \delta v(t), \qquad \xi_4(t) = \frac{\sqrt{2j}}{(1 + U(t)V(t))}\delta V(t), \tag{40}$$

we can construct an anti-symmetric tensor $T_{ik}(t) = \xi_i(t)\xi'_k(t) - \xi'_i(t)\xi_k(t)$, where $\xi_i(t)$ and $\xi'_i(t)$ are independent displacements from the critical trajectory. The tensor T has six independent components, which can be arranged into a new vector defined by $\mathcal{T}^T = (T_{34}(t), T_{23}(t), T_{41}(t), T_{13}(t), T_{42}(t), T_{12}(t))$ whose equation of motion is exactly $\dot{\mathcal{T}} = i\mathcal{A}\mathcal{T}/\hbar$. Putting things together we find that

$$\det[-\delta^2\tilde{F}(\bar{\mathbf{x}})] \equiv \Delta(T) = T_{34}(T)\exp\left\{-\frac{i}{\hbar}\int_0^T \mathbf{H}_+ dt\right\}. \tag{41}$$

Since T_{34} is related to the linearized motion around the critical trajectory, it can be easily written in terms of the tangent matrix or in terms of second derivatives of the action. Working out the details we find

$$\Delta(T) = \frac{(1+U(0)V(0))}{(1+U(T)V(T))} \left[\det \mathrm{M}_{bb}\right] e^{-\frac{i}{\hbar}\int_0^T \mathbf{H}_+ dt}, \tag{42}$$

where M_{bb} is the lower right 2 by 2 block of the tangent matrix in the coordinates χ_i:

$$\mathrm{M}(T) = \begin{pmatrix} \mathrm{M}_{11}(T) & \mathrm{M}_{12}(T) & \mathrm{M}_{13}(T) & \mathrm{M}_{14}(T) \\ \mathrm{M}_{21}(T) & \mathrm{M}_{22}(T) & \mathrm{M}_{23}(T) & \mathrm{M}_{24}(T) \\ \mathrm{M}_{31}(T) & \mathrm{M}_{32}(T) & \mathrm{M}_{33}(T) & \mathrm{M}_{34}(T) \\ \mathrm{M}_{41}(T) & \mathrm{M}_{42}(T) & \mathrm{M}_{43}(T) & \mathrm{M}_{44}(T) \end{pmatrix} \equiv \begin{pmatrix} \mathrm{M}_{aa} & \mathrm{M}_{ab} \\ \mathrm{M}_{ba} & \mathrm{M}_{bb} \end{pmatrix}. \tag{43}$$

Differentiating both sides of Eqs. (26) and (27) and conveniently re-arranging the terms we identify

$$\mathrm{M}_{bb} = \frac{i\hbar}{\frac{\partial^2\mathcal{S}}{\partial u'\partial v''}\frac{\partial^2\mathcal{S}}{\partial U'\partial V''} - \frac{\partial^2\mathcal{S}}{\partial u'\partial V''}\frac{\partial^2\mathcal{S}}{\partial U'\partial v''}} \begin{pmatrix} -\frac{\partial^2\mathcal{S}}{\partial U'\partial V''} & 2j\left(\frac{1}{1+U'V'}\right)^2 \frac{\partial^2\mathcal{S}}{\partial u'\partial V''} \\ \frac{\partial^2\mathcal{S}}{\partial U'\partial v''} & -2j\left(\frac{1}{1+U'V'}\right)^2 \frac{\partial^2\mathcal{S}}{\partial u'\partial v''} \end{pmatrix}. \tag{44}$$

D. The final formula

Collecting all the results for the exponent and pre-factor, the final formula for the semi-classical limit of the canonical-spin coherent state propagator becomes

$$K\left(z''^*, s''^*, z', s', T\right) = \left[\left(\frac{1+U''V''}{1+U'V'}\right)\frac{1}{\det \mathrm{M}_{bb}}\right]^{1/2} \exp\left\{\frac{i}{\hbar}(\mathcal{S}+\mathcal{G}) - \Lambda\right\} \tag{45}$$

where

$$\begin{aligned} \mathcal{S}(z''^*, s''^*, z', s', T) &= \int_0^T \left\{\frac{i\hbar}{2}\left(\dot{u}v - \dot{v}u\right) - i\hbar j\left(\frac{U\dot{V} - V\dot{U}}{1+UV}\right) - \tilde{H}\right\} dt \\ &- \frac{i\hbar}{2}\left(u'v' + u''v''\right) - i\hbar j \ln\left[(1+U'V')(1+U''V'')\right], \end{aligned} \tag{46}$$

$$\begin{aligned} \mathcal{G}(z''^*, s''^*, z', s', T) &= \frac{1}{2}\int_0^T \left\{\frac{\partial^2\tilde{H}}{\partial v\partial u} + \frac{1}{2}\left[\frac{\partial}{\partial V}\frac{(1+VU)^2}{2j}\frac{\partial\tilde{H}}{\partial U} + \frac{\partial}{\partial U}\frac{(1+VU)^2}{2j}\frac{\partial\tilde{H}}{\partial V}\right]\right\} dt \\ &\equiv \frac{1}{2}\int_0^T \mathbf{H}_+ dt. \end{aligned} \tag{47}$$

and

$$\Lambda = \frac{1}{2}\left(|z'|^2 + |z''|^2\right) + j \ln\left[(1+|s'|^2)(1+|s''|^2)\right]. \tag{48}$$

Alternatively, the pre-factor can be written explicitly in terms of derivatives of the action acording to Eq. (44),

$$\left[\left(\frac{1+U''V''}{1+U'V'}\right)\frac{1}{\det \mathrm{M}_{bb}}\right]^{1/2} \rightarrow \left[\frac{(1+U''V'')(1+U'V')}{2j}[\det\Sigma]\right]^{1/2}, \tag{49}$$

where

$$\Sigma = \frac{i}{\hbar}\begin{pmatrix} \frac{\partial^2\mathcal{S}}{\partial u'\partial v''} & \frac{\partial^2\mathcal{S}}{\partial u'\partial V''} \\ \frac{\partial^2\mathcal{S}}{\partial U'\partial v''} & \frac{\partial^2\mathcal{S}}{\partial U'\partial V''} \end{pmatrix}. \tag{50}$$

All quantities are to be calculated at the stationary trajectory (we have removed the bar on top these quantities to simplify the notation).

V. SIMPLE APPLICATIONS

In this section we shall apply the semiclassical formula Eq.(45) to three simple situations: (a) non-interacting spin and field operators; (b) the limit of very large spins and; (c) the case of spin 1/2. In each case we shall see that the propagator obtained from Eq.(45) reduces to well known results.

A. Non-interacting Hamiltonian

If the Hamiltonian can be separated into $\hat{H} = \hat{H}_z + \hat{H}_s$, where $\hat{H}_z = \hat{H}_z(\hat{a}, \hat{a}^\dagger)$ and $\hat{H}_s = \hat{H}_s(\hat{J}_+, \hat{J}_-, \hat{J}_z)$, then $\tilde{H} \equiv \langle z, s|\hat{H}|z, s\rangle = \tilde{H}_z + \tilde{H}_s$, where $\tilde{H}_z \equiv \langle z|\hat{H}_z|z\rangle = \tilde{H}_z(z^*, z)$ and $\tilde{H}_s \equiv \langle s|\hat{H}_s|s\rangle = \tilde{H}_s(s^*, s)$. Therefore, the complex action of Eq. (24) takes the form

$$\mathcal{S}(z''^*, s''^*, z', s', T) = \mathcal{S}_z(z''^*, z', T) + \mathcal{S}_s(s''^*, s', T), \tag{51}$$

where

$$\begin{aligned} \mathcal{S}_z(z''^*, z', T) &= \textstyle\int_0^T \left\{\frac{i\hbar}{2}(\dot{u}v - \dot{v}u) - \tilde{H}_z\right\} dt - \frac{i\hbar}{2}(u'v' + u''v''), \\ \mathcal{S}_s(s''^*, s', T) &= \textstyle\int_0^T \left\{-i\hbar j\left(\frac{U\dot{V} - V\dot{U}}{1+UV}\right) - \tilde{H}_s\right\} dt - i\hbar j \ln\left[(1+U'V')(1+U''V'')\right]. \end{aligned} \tag{52}$$

Moreover, the term $\mathcal{G}$ of Eq. (47) becomes

$$\mathcal{G}(z''^*, s''^*, z', s', T) = \mathcal{G}_z(z''^*, z', T) + \mathcal{G}_s(s''^*, s', T), \tag{53}$$

where

$$\begin{aligned} \mathcal{G}_z(z''^*, z', T) &= \tfrac{1}{2}\int_0^T \tfrac{\partial^2 \tilde{H}}{\partial v \partial u} dt, \\ \mathcal{G}_s(s''^*, s', T) &= \tfrac{1}{4}\int_0^T \left[\tfrac{\partial}{\partial V}\tfrac{(1+UV)^2}{2j}\tfrac{\partial \tilde{H}}{\partial U} + \tfrac{\partial}{\partial U}\tfrac{(1+UV)^2}{2j}\tfrac{\partial \tilde{H}}{\partial V}\right] dt. \end{aligned} \tag{54}$$

Finally, the $\det \Sigma$ simplifies to

$$\det \Sigma = -\frac{1}{\hbar^2}\left[\frac{\partial^2 \mathcal{S}_z}{\partial u' \partial v''}\frac{\partial^2 \mathcal{S}_s}{\partial U' \partial V''}\right]. \tag{55}$$

Therefore, for non-interacting systems, Eq. (45) amounts to

$$K\left(z''^*, s''^*, z', s', T\right) \equiv K_z\left(z''^*, z', T\right) \times K_s\left(s''^*, s', T\right), \tag{56}$$

where

$$\begin{aligned} K_z\left(z''^*, z', T\right) &= \sqrt{\frac{i}{\hbar}\frac{\partial^2 \mathcal{S}_z}{\partial u' \partial v''}}\, e^{\frac{i}{\hbar}(\mathcal{S}_z + \mathcal{G}_z) + \frac{1}{2}|z'|^2 + \frac{1}{2}|z''|^2}, \\ K_s\left(s''^*, s', T\right) &= \sqrt{\frac{i}{\hbar}\frac{(1+U''V'')(1+U'V')}{2j}\frac{\partial^2 \mathcal{S}_s}{\partial U' \partial V''}}\, e^{\frac{i}{\hbar}(\mathcal{S}_s + \mathcal{G}_s) - j\ln\left[(1+|s'|^2)(1+|s''|^2)\right]} \end{aligned} \tag{57}$$

are exactly the semiclassical propagators known in the literature for the Weyl and SU(2) groups respectvely (see, for example, Refs. [14] and [24]).

B. The limit of large spin

Following Perelomov [34], we set $s = w/\sqrt{2j}$, $\hat{J}_+ = \sqrt{2j}\hat{a}^\dagger$ and let $j \to \infty$. In this limit the spin coherent states transform into canonical coherent states:

$$|s\rangle \to |w\rangle = \frac{\exp\left\{w\hat{a}^\dagger\right\}}{\left(1 + \frac{|w|^2/2}{j}\right)^j}|-j\rangle \approx e^{w\hat{a}^\dagger - \frac{1}{2}|w|^2}|0\rangle. \tag{58}$$

In this case, discarding terms smaller than j^{-1} we obtain

$$
\begin{aligned}
& j\frac{s\dot{s}^* - \dot{s}s^*}{1+ss^*} \to \frac{1}{2}(w\dot{w}^* - \dot{w}w^*), \\
& j\ln\left[(1+s's'^*)(1+s''s''^*)\right] \to -\frac{1}{2}\left(w'w'^* + w''w''^*\right), \\
& \frac{\partial}{\partial s^*}\frac{(1+ss^*)^2}{2j}\frac{\partial \tilde{H}}{\partial s} + \frac{\partial}{\partial s}\frac{(1+ss^*)^2}{2j}\frac{\partial \tilde{H}}{\partial s^*} \to 2\frac{\partial^2 \tilde{H}}{\partial w \partial w^*}
\end{aligned}
\tag{59}
$$

and

$$
\tfrac{(1+s''s''^*)(1+s's'^*)}{2j}\det\begin{pmatrix} \frac{i}{\hbar}\frac{\partial^2 \mathcal{S}}{\partial z'\partial z''^*} & \frac{i}{\hbar}\frac{\partial^2 \mathcal{S}}{\partial z'\partial s''^*} \\ \frac{i}{\hbar}\frac{\partial^2 \mathcal{S}}{\partial s'\partial z''^*} & \frac{i}{\hbar}\frac{\partial^2 \mathcal{S}}{\partial s'\partial s''^*} \end{pmatrix} \to \det\begin{pmatrix} \frac{i}{\hbar}\frac{\partial^2 \mathcal{S}}{\partial z'\partial z''^*} & \frac{i}{\hbar}\frac{\partial^2 \mathcal{S}}{\partial z'\partial s''^*} \\ \frac{i}{\hbar}\frac{\partial^2 \mathcal{S}}{\partial w'\partial z''^*} & \frac{i}{\hbar}\frac{\partial^2 \mathcal{S}}{\partial w'\partial w''^*} \end{pmatrix}. \tag{60}
$$

Equations (59) to (60) applied to Eqs. (45) and (49) produces the two-dimensional semiclassical propagator for canonical coherent states [15].

C. Spin-$\frac{1}{2}$ Systems

The semiclassical approximation developed in sections II to IV employed explicitly the limit $j \to \infty$. In this subsection we discuss the application of Eq. (45) to spin-$\frac{1}{2}$ systems, whose general Hamiltonian is given by

$$
\hat{H} = \hat{H}_0 + \hat{H}_s \equiv \hat{H}_0(\hat{a}, \hat{a}^\dagger) + \hbar\hat{\mathbf{s}}\cdot\hat{\mathbf{C}}(\hat{a}, \hat{a}^\dagger). \tag{61}
$$

In this case the classical Hamiltonian reads

$$
\tilde{H}(u,v,U,V) \;=\; \langle z|\tilde{H}_0|z\rangle + \hbar\langle s|\hat{\mathbf{s}}|s\rangle\cdot\langle z|\hat{\mathbf{C}}|z\rangle = \tilde{H}_0(u,v) + \frac{\hbar}{2}\tilde{H}_s(u,v,U,V), \tag{62}
$$

where

$$
\tilde{H}_s(u,v,U,V) = C_1(u,v)\frac{U+V}{1+UV} - iC_2(u,v)\frac{V-U}{1+UV} - C_3(u,v)\frac{1-UV}{1+UV}. \tag{63}
$$

and $\langle z|\hat{\mathbf{C}}|z\rangle \equiv (C_1(u,v), C_2(u,v), C_3(u,v))$.

The equations of motion are given explicitly by

$$
\begin{aligned}
\dot{v} &= \tfrac{i}{\hbar}\tfrac{\partial}{\partial u}(\tilde{H}_0 + \tfrac{\hbar}{2}\tilde{H}_s), \\
\dot{u} &= -\tfrac{i}{\hbar}\tfrac{\partial}{\partial v}(\tilde{H}_0 + \tfrac{\hbar}{2}\tilde{H}_s), \\
\dot{V} &= \tfrac{i}{2}\left[(C_1+iC_2) - (C_1-iC_2)V^2 + 2C_3V\right], \\
\dot{U} &= \tfrac{i}{2}\left[(C_1+iC_2)U^2 - (C_1-iC_2) - 2C_3U\right].
\end{aligned}
\tag{64}
$$

In the limit of small $\hbar$ we can drop the terms $\frac{\hbar}{2}\tilde{H}_s$ on the first two equations and decouple u and v from the spin variables U and V. These, on the other hand, describe the precession of the spin in the external field $\mathbf{C}(u, v)$ generated by the orbital motion. In this approximation the orbital part of the action also separates from the total action and the semiclassical propagator can be written as

$$K\left(z''^*, s''^*, z', s', T\right) = K_z\left(z''^*, z', T\right) \times K_{s[u,v]}\left(s''^*, s', T\right), \tag{65}$$

where K_z is the one-dimensional canonical propagator and $K_{s[u,v]}$ can be written as [23]

$$K_{s[u,v]}\left(s''^*, s', T\right) = \frac{a^*(t) - b^*(t)s' + b(t)s''^* + a(t)s''^* s'}{(1 + |s''|^2)(1 + |s'|^2)}. \tag{66}$$

The coefficients $a(t)$ and $b(t)$ are obtained from the differential equation

$$\frac{\mathrm{d}W}{\mathrm{d}t} = -\frac{i}{2}\sigma \cdot \mathbf{C}(t) W(t) \tag{67}$$

where

$$W(t) = \begin{pmatrix} a(t) & b(t) \\ -b^*(t) & a^*(t) \end{pmatrix}, \tag{68}$$

σ are the Pauli matrices and $W(0) = \mathbf{1}$. Since Eq.(66) is the exact propagator for a spin in an external field, Eq.(65) can also be derived directly from the path integral approach

$$K\left(z''^*, s''^*, z', s', T\right) = \int \frac{\mathcal{D}[u]\mathcal{D}[v]}{\pi} K_{s[u,v]}\left(s''^*, s', T\right) e^{\frac{i}{\hbar} F_{z0}[u,v,T]} \tag{69}$$

where the steepest descent approximation is performed only in the orbital action F_{z0} (which contains only H_0). The spin propagator K_s is viewed as a slow varying pre-factor and is simply calculated at the stationary trajectory [32, 33]. This shows that the semiclassical formula Eq. (45) can also be used for systems with spin $j = 1/2$, in spite of the large spin limit considered in its derivation.

Acknowledgments

This work was partly supported by FAPESP and CNPq. ADR and MAMA thank Dr. Marcel Novaes and Dr. Fernando Parisio for helpful discussions.

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